Math 249 Lecture 1 Notes

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1 The Symmetric Group

Definition 1.1. Let A be a set. A *permutation* is a bijection $\sigma : A \to A$.

1.1 Basic facts

- Permutations form a set $S(A)$, which acts on the set A. We notate this as $S(A) \circlearrowright A$.
- If $A \cong B$, then $S(A) \cong S(B)$. Then we may think of $S(\cdot)$ as a functor $(\mathbf{Set}, \cong) \to \mathbf{Grp}$.
- If A is finite, we usually look at $[n] := \{1, \dots, n\}$. Then we call $S_n := S([n])$.
- There are different notations for permutations:
  - 2-row:
  $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix},$$
  - 1-row: $(\sigma(1), \dots, \sigma(n))$
  $$\sigma = \begin{pmatrix} 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix},$$
  - Cycle notation:
  $$\sigma = [1\ 5\ 2][4\ 6].$$
  Cycle notation is useful because to find inverses, we just reverse the cycles. For example,
  $$\sigma^{-1} = [1\ 2\ 5][4\ 6].$$
  Moreover, the order of σ is easy to compute.
  $$\operatorname{ord}(\sigma) = \operatorname{lcm}(\text{cycle lengths}).$$

1.2 Some counting with S_n

$|S_n| = n!$, and for every $\sigma \in S_n$, the lengths of the cycles of σ form a partition of n. For example, using σ above, we have the partition $6 = 3 + 2 + 1$ (the 1 is the implicit cycle (3)).

1.2.1 Counting sizes of conjugacy classes of S_n

What happens if you conjugate by a permutation? Suppose $a \xrightarrow{\sigma} b$. Then $\tau\sigma\tau^{-1}$ sends $\tau(a)$ to $\tau(b)$. So the cycle decomposition of $\tau\sigma\tau^{-1}$ is

$$\tau\sigma\tau^{-1} = [\tau(a_1)\,\tau(a_2)\,\cdots\,\tau(a_k)], \qquad \text{where } \sigma = [a_1\,a_2\,\cdots\,a_k].$$

How large are the conjugacy classes C_λ? Make a "template" of the cycle lengths: e.g. $\lambda \leftarrow (5, 2, 2, 2, 1, 1)$. There are $n!$ ways to fill it, but it counts each $\sigma \in C_\lambda$ many times. So say $\lambda \leftarrow (1^{r_1}, 2^{r_2}, \dots)$. Factor $\prod_j r_j!$ for swapping cycles, and factor $\prod_i \lambda_i = \prod_j j^{r_j}$ for rotating cycles. Then

$$|C_\lambda| = n!/z_\lambda, \qquad \text{where } z_\lambda = \prod_j j^{r_j} r_j!.$$

Now $z_\lambda = |Z(\lambda)|$, and $Z(\lambda) \cong (S_{r_1} \times S_{r_2} \times \cdots) \rtimes (C_1^{r_1} \times C_2^{r_2} \times \cdots)$; these terminate when we run out of r_j. This is an example of a "wreath product."

1.2.2 Counting k-subsets of $[n]$

Define

$$\binom{n}{k} := \text{number of } k\text{-subsets of } [n], \qquad \binom{A}{k} := \{S \subseteq A : |S| = k\}.$$

Then S_n acts transitively on $\binom{[n]}{k}$, so $\binom{n}{k} = n!/|\mathrm{Stab}([k])|$. Then note that $\mathrm{Stab}([k]) \cong S_k \times S_{n-k}$. So $\binom{n}{k} = n!/(k!(n-k)!)$.

Alternatively, count words on $\omega_1, \dots, \omega_k$ from $[n]$, with distinct letters ("k-permutation"):

$$[n]_k := n(n-1)(n-2)\cdots(n-k+1).$$

This gives each subset $\{\omega_1, \dots, \omega_k\}$ $k!$ times. So

$$\binom{n}{k} = \frac{[n]_k}{k!} = \frac{n!}{k!(n-k)!}.$$

In general, this works more generally for integers, fractions, etc.:

$$[\alpha]_k = \alpha(\alpha-1)\cdots(\alpha-k+1)$$

$$\binom{\alpha}{k} = \frac{[\alpha]_k}{k!},$$

which is called Newton's binomial coefficient.